Proc. R. Soc. Lond. A **420**, 81–91 (1988)
Printed in Great Britain

The parametrically maintained Foucault pendulum and its perturbations

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(*Received* 19 *April* 1988)

A Foucault pendulum may be maintained indefinitely by parametric excitation, but as a consequence of imperfections in its construction, seismic noise and other perturbations it is liable to precess at other than the ideal rate. It has been suggested by Braginsky *et al.* (*Phys. Rev. Lett.* **53**, 863 (1984)) that the disturbing factors might be reduced to such an extent that the Thirring–Lense effect of General Relativity could be verified. The present analysis reveals serious problems in addition to those they considered, but indicates how parametric excitation helps to overcome them. Nevertheless their assessment of the chance of success reflects an optimism that the present author cannot share.

Introduction

The Foucault pendulum has been the topic of so many reports since its demonstration (Foucault 1851) attracted the cultivated and the fashionable of Paris, that its interest might be thought exhausted. Nevertheless the demands of museums and other public places have continued to stimulate ingenious new methods of maintaining the oscillations without disturbing the precessional rate, and it has further been suggested (Braginsky, Polnarev & Thorne 1984; henceforth B.P.T.) that it might not be out of the question to verify by its means the Thirring–Lense (1918; T.L.) effect predicted by General Relativity. Both these motivations underlie the work described here, which extends the searching analysis by B.P.T. of the potential sources of error. Little will be said about a model constructed by the author to test the behaviour of a parametrically maintained pendulum, except in illustration of theoretical points. No model built yet has achieved anything approaching the precision demanded by the T.L. effect. This effect, referred to by B.P.T. as the gravitomagnetic effect, is a dragging round of the inertial frame with respect to that defined by the distant stars, as a result of the Earth's rotation; theoretically it amounts to 0.22″ per year, i.e. 6×10^{-10} of the Earth's rotational speed. There is an enormous hiatus between a convincing museum demonstration of the Earth's rotation and a test of General Relativity.

The perennial bugbear of Foucault pendulums is their tendency to develop an elliptical orbit, which, as a consequence of anharmonicity of the restoring force, gives rise to an intrinsic precession that may easily overwhelm the Foucault precession. The popular way of controlling ellipticity is the Charron ring (1931; but see also Whittle 1887). Although effective, its demerit is that it relies on friction, an unacceptable feature if high precision is demanded. Parametric excitation also acts against the development of ellipticity, without disturbing the Foucault precession. It has been used before (Whittle 1887; Baker 1932) but not, it seems, analysed in any detail.

The support from which the pendulum is hung is vibrated in a vertical direction at twice the pendulum frequency, with the phase controlled so that the support rises at its maximum speed as the pendulum passes through the centre of its swing, when the tension of the wire is greatest; it falls at its maximum speed at the ends of the swing when the tension is least. Thus there is a steady input of energy to compensate dissipative losses. If the pendulum develops an elliptical trajectory, the vibration normal to the principal motion is in phase quadrature with it, and is attenuated rather than amplified by the parametric process (Pippard 1978). The minor axis, and the consequent unwanted precession, are in this way reduced considerably below what they would be for the uncontrolled pendulum.

An imperfect Foucault pendulum must approximate closely to an ideal spherical pendulum, executing simple harmonic motion, if it is to work even moderately well. This allows the imperfections to be discussed by a peculiarly simple perturbation treatment. For a perfect harmonic oscillator has the unique property that the action of a number of perturbing forces is the sum of their actions as if each alone were present. This means that the effect of each imperfection can be treated separately; if they are small enough that first-order processes are all that need be considered, even imperfections that produce nonlinearity can be so included. The first stage of the analysis is therefore to establish a rather general method of treating imperfections, which will then be applied to a number of special cases.

Perturbation of an isotropic two-dimensional harmonic vibrator

An ideal loss-free vibrator, of mass m, subject to a synchronized perturbing force in its plane, is systematically affected only by the fundamental component. Let it initially move clockwise in a narrow elliptical orbit with the major axis along x, so that its unperturbed coordinates (x_0, y_0) vary as

$$x_0 = a\cos\omega t \quad \text{and} \quad y_0 = -b\sin\omega t\,(b \ll a), \tag{1}$$

and let the fundamental component of the perturbing force be resolved into components as follows:

$$F_{ac}\cos\omega t + F_{as}\sin\omega t \quad \text{along the major axis,}$$

and

$$F_{bc}\cos\omega t + F_{bs}\sin\omega t \quad \text{along the minor axis.}$$

After one cycle the perturbations of the motion have the form

$$\Delta x_1 = (\pi/m\omega^2)(F_{ac}\sin\omega t - F_{as}\cos\omega t)$$

and

$$\Delta y_1 = (\pi/m\omega^2)(F_{bc}\sin\omega t - F_{bs}\cos\omega t).$$

When these perturbations are added to x_0 and y_0 the resulting components along the x and y axes are no longer in phase-quadrature. To determine the orientation of the new ellipse, rotate the axes clockwise through a small angle η, such that the resulting x and y components are once more in quadrature. The rotation produces changes, to first order in η,

$$\Delta x_2 = \eta b \sin \omega t \quad \text{and} \quad \Delta y_2 = \eta a \cos \omega t.$$

The perturbed oscillation is described, relative to the new axes, by

$$\left.\begin{aligned} x &= (a - \lambda F_{as}) \cos \omega t + (\lambda F_{ac} + \eta b) \sin \omega t \\ \text{and} \qquad y &= -(b - \lambda F_{bc}) \sin \omega t + (\eta a - \lambda F_{bs}) \cos \omega t, \quad \text{where} \quad \lambda = \pi / m\omega^2. \end{aligned}\right\} \tag{2}$$

If these are to be in quadrature,

$$(a - \lambda F_{as})(\eta a - \lambda F_{bs}) = (\lambda F_{ac} + \eta b)(b - \lambda F_{bc}), \quad \text{i.e.} \quad \eta \approx \lambda (a F_{bs} + b F_{ac}) / a^2,$$

after neglect of second-order terms. This is the angle through which the principal axis is caused to precess in one cycle of oscillation. The clockwise precessional angular velocity Ω_p may therefore be written

$$\Omega_p \approx (a F_{bs} + b F_{ac}) / 2 m \omega a^2 = (F_{bs} + \epsilon F_{ac}) / 2 m \omega a, \tag{3}$$

where ϵ is the ellipticity, b/a.

The development of ϵ also follows from (2):

$$\dot{\epsilon} = (a \dot{b} - b \dot{a}) / a^2 = -(F_{bc} - \epsilon F_{as}) / 2 m \omega a. \tag{4}$$

The imperfect maintained Foucault pendulum

The Coriolis force responsible for Foucault precession is normal to, and in phase with, the velocity of the pendulum. It is therefore described by $F_{bs} = 2m\Omega\omega a$, where Ω is the vertical component of the Earth's angular velocity at the latitude of the experiment. From (3) it is responsible for the ideal precessional rate $\Omega_p = \Omega$, which is clockwise (positive) in the Northern Hemisphere. The pendulum is, however, never perfectly isotropic and the resulting elliptical motion with its consequent precession must be allowed for. In addition, damping due to air resistance and other dissipative effects must be compensated by parametric amplification. All these processes must be included in the equation of motion, and will be derived separately.

(*a*) *Anisotropy*

In the absence of the Coriolis force the pendulum has two orthogonal principal planes, fixed in the laboratory, in which linear vibrations persist. If the plane of slower vibration, at frequency $\omega - \delta$, is taken as the reference plane from which the orientation θ of the major axis is measured clockwise, and if the faster vibration has frequency $\omega + \delta$, the restoring force coefficients along the principal axes are $m\omega^2(1 \pm 2\delta/\omega)$. The force components along the principal axes are

along the slow axis, $\quad -m\omega^2(1 - 2\delta/\omega)(a \cos\theta \cos\omega t - b \sin\theta \sin\omega t)$;

along the fast axis, $\quad m\omega^2(1 + 2\delta/\omega)(b \cos\theta \sin\omega t + a \sin\theta \cos\omega t)$.

Hence, after resolving along the principal axes of the ellipse, the perturbing forces take the form

$$F_{ac} = 2m\omega\delta a \cos 2\theta, \quad F_{as} = -2m\omega\delta b \sin 2\theta,$$

$$F_{bc} = 2m\omega\delta a \sin 2\theta, \quad F_{bs} = 2m\omega\delta b \cos 2\theta.$$

It follows from (3) and (4) that

$$\Omega_{\mathrm{p}} = 2\delta\epsilon \cos 2\theta \quad \text{and} \quad \dot{\epsilon} \approx -\delta \sin 2\theta. \tag{5}$$

(*b*) *Anharmonicity*

The projection of the pendulum bob onto a horizontal plane behaves as an anharmonic two-dimensional vibrator. Let the excitation energy be E and the momentary horizontal displacement r so that, to second order in r, the kinetic energy $\frac{1}{2}mv^2$ is $E-mgr^2/2l$, l being the length of the pendulum and v the velocity of the bob itself, independent of its direction of motion. The centripetal acceleration v^2/l, towards the point of suspension, is caused by the difference between T, the tension in the wire, and $mg(1-r^2/2l^2)$, the component of weight acting along the wire, also to second order in r. Therefore

$$T = mv^2/l + mg(1-r^2/2l^2) \approx mg + 2E/l - 3mgr^2/2l^2. \tag{6}$$

The first two terms generate a harmonic restoring force in the horizontal plane and the last an anharmonic central repulsive force $F' = (3mgr^2/2l^3)\,\boldsymbol{r}$. For motion according to (1) this force has components

$$F' = (3mg/2l^3)\,(a^2 \cos^2 \omega t + b^2 \sin^2 \omega t)\,(a \cos \omega t, \; -b \sin \omega t),$$

of which only the fundamental Fourier coefficients concern us. Thus

$$F_{ac} = (3mg/2l^3)\,(\tfrac{3}{4}a^3 + \tfrac{1}{4}ab^2)$$

and

$$F_{bs} = -(3mg/2l^3)\,(\tfrac{1}{4}a^2 b + \tfrac{3}{4}b^3),$$

from which the standard expression (Olsson 1978) follows, by use of (3), with $g/l = \omega^2$,

$$\Omega_{\mathrm{p}} \approx c\epsilon, \quad \text{where } c = 3\omega a^2/8l^2, \tag{7}$$

and $\dot{\epsilon} = 0$. The sense of precession is the same as that of the orbital motion.

(*c*) *Dissipation*

In the steady state, a is maintained very nearly constant by parametric amplification, whereas b is always small. The dissipation, even if nonlinear, may be represented well enough by introducing separate effective relaxation times for the major and minor axes, such that without amplification

$$\dot{a} = -a/\tau_a \quad \text{and} \quad \dot{b} = -b/\tau_b. \tag{8}$$

(*d*) *Parametric amplification*

Let the pendulum support vibrate vertically at twice the pendulum frequency, with upward displacement $-\beta \sin(2\omega t + \phi)$. Ideally ϕ is made to vanish, but it is retained so that the effect of maladjustment can be estimated. If the pendulum displacement is small the vertical acceleration of the support, carrying the bob with it, increases the tension in the wire by $4m\omega^2\beta \sin(2\omega t+\phi)$. The resultant horizontal perturbing force is central, with components

$$F' = (4m\omega^2\beta/l)\sin(2\omega t+\phi)(-a\cos\omega t, b\sin\omega t).$$

As with the anharmonic perturbation, only the fundamental component concerns us:

$$F'_{ac} = Aa\sin\phi,\ F'_{as} = Aa\cos\phi,$$

$$F'_{bc} = -Ab\cos\phi,\ F'_{bs} = Ab\sin\phi,$$

where $A = 2m\omega^2\beta/l$. Hence, from (3) and (4)

$$\Omega_{\text{p}} = -(2\omega\beta\epsilon/l)\sin\phi \tag{9}$$

and

$$\dot{\epsilon} = -(2\omega\beta\epsilon/l)\cos\phi. \tag{10}$$

The decrement of ϵ, with effective time constant τ_{p} such that $1/\tau_{\text{p}} = (2\omega\beta/l)\cos\phi$, is the resultant of a decrement of b and an increment of a, both with time constant $2\tau_{\text{p}}$. If the amplification is adjusted, as assumed in what follows, so that $\phi = 0$ and $\omega\beta/l = 1/\tau_a$, the major axis is maintained at constant length while the minor axis suffers decrement at the enhanced rate. The effective time constant for b, and hence for ϵ is τ', where

$$1/\tau' = 1/\tau_a + 1/\tau_b. \tag{11}$$

When $\phi = 0$, or for all ϕ if $\epsilon = 0$, (9) shows that parametric amplification does not affect the rate of precession. At the worst, maladjustment of ϕ only modifies to a minor extent any existing error in precessional rate.

Equation of motion

The results expressed by (5), (7), (9) and (11) may be combined into equations of motion for θ and ϵ, $\dot{\theta}$ being the sum of the basic Foucault precession rate Ω and all perturbations Ω_{p}:

$$\dot{\theta} = \Omega + \epsilon(2\delta\cos 2\theta + c), \tag{12}$$

and

$$\dot{\epsilon} = -\delta\sin 2\theta - \epsilon/\tau'. \tag{13}$$

It follows that

$$\mathrm{d}\epsilon/\mathrm{d}\theta = -(\mu\sin 2\theta + \gamma\epsilon)/(1 + 2\mu\epsilon\cos 2\theta + \alpha\epsilon), \tag{14}$$

where

$$\mu = \delta/\Omega, \alpha = c/\Omega \quad \text{and} \quad \gamma = 1/\Omega\tau'.$$

Only in a very poorly adjusted pendulum will the precession rates due to anisotropy (as represented by μ) and anharmonicity (α) cause a major disturbance to the Foucault precession, and it should be a good first approximation to put the denominator in (14) equal to unity when determining how ϵ varies with θ. If necessary, successive approximations can then be evaluated, but this will not be done here. With this simplification the steady-state solution of (14) can be written down immediately:

$$\epsilon = [-\mu/(4+\gamma^2)^{\frac{1}{2}}]\sin(2\theta - \zeta), \quad \text{where} \quad \tan\zeta = 2/\gamma = 2\Omega\tau'. \tag{15}$$

The advantage of parametric amplification is clear in the coefficient defining the magnitude of ϵ. A pendulum at the latitude of Cambridge makes one revolution in 30.29 h; in the model τ' is 40 min, $\gamma \sim 7.2$ and the maximum size of the minor axis is, according to (15), less by a factor of 3.7 than without amplification.

When (15) is substituted in (12), rearrangement gives

$$\dot{\theta}/\Omega = 1+2\mu^2/(4+\gamma^2)+\alpha\mu\sin(2\theta-\zeta)/(4+\gamma^2)^{\frac{1}{2}}-\mu^2\sin(4\theta-\zeta)/(4+\gamma^2)^{\frac{1}{2}}, \quad (16)$$

of which the last term may be neglected without great error, because μ is normally much less than α. The precession rate is slightly increased by the second term and modulated by the third. Now a modulated rotational speed of the form $\Omega_1+\Omega_2\sin(n\theta)$ gives rise to a mean speed of $(\Omega_1^2-\Omega_2^2)^{\frac{1}{2}}$. To second order in μ, then, which is as much as is justified by the approximations made so far, the mean rate of precession, $\bar{\Omega}$, follows from (16):

$$\bar{\Omega}/\Omega = 1+2\mu^2(1-\tfrac{1}{4}\alpha^2)/(4+\gamma^2). \quad (17)$$

The anisotropy, represented by μ, has a primary consequence of increasing $\bar{\Omega}$. This can be understood by analogy with a pair of coupled vibrators, whose normal modes are closest in frequency, for a given coupling constant, when the vibrators are exactly tuned together: mistuning separates the modes and increases their beat frequency, of which the precessional rate is the analogue here. The secondary effect of μ is to modulate the speed through the anharmonicity, and this always decreases $\bar{\Omega}$. In the present case the two effects cancel when $\alpha = 2$, or when $a/l = 4/(3N)^{\frac{1}{2}}$, N being the number of complete vibrations of the pendulum during one cycle of precession. In the model $l = 8.5$ m, $N \sim 18700$ and thus $a \sim 14$ cm for cancellation.

At the larger amplitudes usual for demonstration pendulums the anharmonicity easily dominates, having an effect proportional to a^4, and the precession rate is reduced. For example, at an operating amplitude $a = \frac{1}{25}l$, $\frac{1}{4}\alpha^2 = 31.5$, so that $\bar{\Omega}/\Omega \sim 1-1.1\mu^2$. For the error to be less than 1%, μ must be less than 0.1 and δ less than $5\times10^{-6}\omega$. Because $\omega \propto l^{-\frac{1}{2}}$, the difference between the values of l in the principal planes must be less than $2\times10^{-5}l$, or 0.17 mm, which does not seem an excessively severe demand.

To study the T.L. effect, however, an improvement over this by at least a factor of 10^{10} is needed, and l must be the same in the principal planes to better than 1.7 nm. This poses a very difficult problem, not perhaps to provide adjustments that enable high isotropy to be achieved, but because of the difficulty of maintaining the adjustment over the prolonged period needed for a measurement. This is but one of the obstacles to achieving the precision required for T.L., many of which have been outlined by B.P.T. In the following section some of these are examined in somewhat more detail, and the catalogue further enlarged.

Sources of error

(a) *The suspension*

If we leave aside, as too-little developed for assessment, superconducting levitation of the pivot, the available suspensions are of three types: needle or ball, gimbal and flexible rod. With needles and small balls the contact area is apt to wear, and the use of gimbals can spread the load, especially with the crossed cylinder arrangement of Longden (1919). It is almost impossible, however, to avoid elastic hysteresis and rolling friction in any of these arrangements. Consequently no potential function can be defined for the pendulum and the lagrangian theory of normal modes is inapplicable. It is not difficult to show (as will not be done here) that hysteresis in a gimbal mounting causes extra precession of alternating sign in successive octants of the precession cycle, and this, as in (16), perturbs Ω. In addition to this, in a gimbal one pair of pivots, or knife edges, supports the anvils on which the second pair rest. Any deformation of the second pair is likely to change the effective pendulum length for vibration in both planes, but deformation of the first pair changes the length only for its own plane of vibration. For this reason gimbals cannot be relied on to maintain their initial fine tuning. It is indeed unlikely that any mechanical pivot can be devised to meet the stringent conditions demanded by T.L.

The materials problem presented by a flexing wire support is well known to any who have attempted to reproduce Foucault's original demonstration, hanging a heavy bob on a steel wire that is clamped or soldered to a boss at the top; it is not long, usually, before fatigue causes the wire to break at its point of emergence from the boss. This is an extreme example of the conditions endured by a rod that is longitudinally loaded while each filament is subjected to a pattern of severe alternating strain: a pattern most conducive to creep. The severity of the strain is reduced, as indicated below, by using a thicker wire or rod as the flexing member, but one must do more than avoid the risk of fracture; if creep occurs, it cannot be relied on to maintain the isotropy of the suspension, and one should aim at choosing the material and dimensions so that creep causes no change of pendulum frequency as great as the allowed value of δ in (5).

The theory of flexure of a rod with one fixed end, under a steady tension and a transverse force at its free end, is well known. If the rod is infinitely long it remains straight except near the top ($\xi = 0$). The lateral displacement at a point ξ below this has the form $C(\lambda\xi + e^{-\lambda\xi} - 1)$ where C is a constant and $\lambda^2 = 4mg/\pi E\rho^4$, E being Young's modulus and ρ the radius of the rod. The maximum curvature is $C\lambda^2$ and the maximum strain of any filament $C\lambda^2\rho$. At the furthest extent of the pendulum's displacement ($\xi = l$), $C\lambda l \approx a$ if $\lambda l \gg 1$, and the maximum strain is $(2a/\rho l)(mg/\pi E)^{\frac{1}{2}}$. Although no data are available that relate to the extremely slow creep rates that are tolerable, one may guess from published curves (Frost & Ashby 1982) that a strain of 10^{-4} could be permitted in tungsten. The rod diameter for a pendulum 8.5 m long, when the amplitude is the optimal 14 cm, must be not less than $1.8m^{\frac{1}{2}}$ mm (m in kilograms), and the length needed is several times λ^{-1}, which for this diameter is $0.27m^{\frac{1}{2}}$ m. Provided the bob is not more than a few kilograms in mass the requirements can be met. An oriented carbon-fibre composite might be even better than tungsten. Operating the pendulum in a high vacuum, as demanded by other considerations, should help to prevent ageing effects.

(*b*) *Dissipation*

B.P.T.'s analysis is based on the use of a free pendulum *in vacuo*, for which a decay time as large as 1 year has been demonstrated. This places especially great demands on isotropy because the factor of improvement, $1 + 1/4\gamma^2$, in (17), conferred by parametric amplification is lacking. There is no simple way, however, of capturing this advantage without disturbing the precession rate. One might, for example, use air damping to enhance γ, but the air will be at rest in the laboratory frame. As seen by an outside observer, an ideal pendulum at one of the Poles swings in an unchanging plane while the air rotates once in a day. Let us examine the behaviour of a free pendulum in a viscous fluid rotating at angular velocity $-\Omega$, on the assumption that the drag force is $-k\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the bob. In addition to the damping of the vibration, with time constant $\tau_d = 2m/k$, there is a lateral force which, for a vibration described by (1), has components

$$F_{as} = k\Omega b \quad \text{and} \quad F_{bc} = k\Omega a. \tag{18}$$

This produces no precession but, as (4) shows, causes the axial ratio to change at a rate

$$\dot{\epsilon} \approx -k\Omega/2m\omega = -\Omega/\omega\tau_d. \tag{19}$$

Anharmonicity then introduces precession as a secondary consequence, and according to (7),

$$\dot{\Omega}_p = -3\Omega a^2/8l^2\tau_d. \tag{20}$$

Because the T.L. precession is $6 \times 10^{-10}\Omega$, a free 8.5 m pendulum with the optimal a/l ratio of $\frac{1}{60}$ needs $\tau_d > 1.7 \times 10^5$ years if the spurious precession is not to overtake it in the course of a year. In principle the effect may be allowed for, but the precision to which τ_d would need to be known is formidable.

Parametric amplification, although it controls the growth of the minor axis, gives us no significant amelioration, because it depends on τ_d being fairly short. It causes the growth given by (20) to saturate after about time τ_d at a value rather less than $3\Omega a^2/8l^2$, say 10^5 times the T.L. rate.

Because we have here a most serious obstacle to the measurement it is worth considering how it might be countered. Turning the dissipative medium, including the suspension (which provides some dissipation, albeit small) in synchronism with the pendulum's precession has already been discussed by B.P.T., and rejected because it allows the accumulation of effects due to anisotropy. In principle, however, the medium may be rocked back and forth through a very small angle at twice the pendulum frequency so as to stay in the same mean position, thus avoiding cumulative errors, yet produce no dissipative expansion of the minor axis. Thus, if Ω in (18) is replaced by $\Omega(1-2\cos 2\omega t)$ the significant perturbation, F_{bc}, has no fundamental Fourier coefficient. In laboratory coordinates the dissipative medium must be carried forward at twice the Foucault angular velocity when the pendulum is at the ends of its swing, and backward at the same rate as it passes through the centre. The linear displacement at radius r needed for compensation by this means is $(r/N)\sin(2\omega t)$, less than $\frac{1}{10}$ mm at a radius of 1 m, but possibly capable of being adjusted more precisely than τ_d can be measured.

The exact magnitude of rocking needed depends on how the retarding force is related to the relative velocity of the medium and the bob, which may not be strictly proportional as assumed in this analysis. By evacuating the pendulum chamber, however, and providing the bob with a magnet that excites Foucault eddy currents in an underlying conducting sheet, strictly linear dissipation may be achieved. One may improve on this by adopting the ingenious device (Mastner *et al.* 1984) of cutting a circular hole in the conducting sheet, so that eddy current drag is confined to the intervals when the pendulum is near its greatest displacement. Then motion along the major axis is slow and is dissipated considerably less than motion along the minor axis which is at its maximum. This highly dependable mechanism enhances the benefits of parametric excitation in keeping b small. The rocking motion appropriate here is one that follows the Foucault precession exactly while the bob lies over the conductor, and reverses the displacement, as rapidly as desired, while it goes through the centre, well away from the conductor.

If the effective radius of the hole is r_h, so that there is no drag when $x < r_h$ and full drag when $x > r_h$, evaluation of the fundamental Fourier coefficient of the drag force shows that the attenuation of motion in the x-direction, compared with that for a complete sheet, is multiplied by $\vartheta - \frac{1}{2}\sin 2\vartheta$, where $\cos\vartheta = r_h/a$; the corresponding factor for motion in the y-direction is $\vartheta + \frac{1}{2}\sin 2\vartheta$. The time-constants from this cause alone are thus in the ratio

$$\tau_b/\tau_a = (\vartheta - \tfrac{1}{2}\sin 2\vartheta)/(\vartheta + \tfrac{1}{2}\sin 2\vartheta). \tag{21}$$

Given adequate control of the pendulum amplitude, one should be able to maintain r_h/a at 0.95, so that $\tau_b/\tau_a \approx \frac{1}{30}$. With modern permanent magnets swinging a few millimetres above the thick copper plate, τ_b can be reduced to 1–2 min without making excessive demands on parametric amplification. Then the residual dissipation, not following the Foucault precession, does not cause steady growth of ϵ according to (19), but instead ϵ saturates at a value $-\Omega\tau_b/\omega\tau_{res}$; τ_{res} here is the time-constant for the residual dissipation alone. From (7) the secondary precession is then seen to be given by

$$\Omega_p/\Omega = \tfrac{3}{8}a^2\tau_b/l^2\tau_{res},$$

equal to the T.L. rate if $a/l = 1.6\times10^{-2}$ (the optimum for $l = 8.5$ m) and $\tau_b = 6\times10^{-6}\tau_d$. With $\tau_b = 3$ min and $\tau_{res} = 1$ year this is achieved.

To reach this condition, the minor axis of the elliptical orbit must be maintained at less than $\frac{1}{2}$Å,† which is a rather more severe demand than B.P.T. contemplate, but may be within the bounds of possibility to detect and correct. It may be remarked that altering the oscillation amplitude of the conductive sheet provides an easier mechanism than their suggested use of adjustable gravitational forces for maintaining ϵ below the permissible limit. Nevertheless, to rock the conductive sheet at precisely the right amplitude is a daunting technical challenge.

(*c*) *Noise*

B.P.T. quote figures for seismic activity at the South Pole that show that even at this exceptionally quiet spot it would be necessary to provide efficient anti-seismic mounting if the T.L. precession were not to be swamped by random drift. It should be noted that their expression includes only the direct effect, and neglects secondary precession resulting from any orbital ellipticity that noise may produce. The ratio of these effects is easily calculated by considering two possible

† 1 Å = 10^{-10} m = 10^{-1} nm.

disturbances to a pendulum with $\epsilon = 0$, by an impulsive force applied normal to the line of motion. If an impulse P strikes the bob as it passes through the centre a transverse velocity $v_y = P/m$ is added to the principal velocity $v_x = \omega a$, turning the plane of vibration through $\delta\phi_1 = P/m\omega a$; this is the direct effect. If, however, the impulse strikes the bob at the end of its swing the x and y motions are in quadrature and the orbit is rendered elliptical without being turned. Because $P/m\omega a$ now expresses the initial value of ϵ the pendulum precesses according to (7) for so long as it takes ϵ to decay, i.e. for time τ' as in (11), and the total precession angle resulting from this impulse is $\delta\phi_2$, where

$$\delta\phi_2 = c\epsilon\tau' = (3\omega\tau' a^2/8l^2)\,\delta\phi_1. \tag{22}$$

For the 2.3 m free pendulum considered by B.P.T., with τ' so long that it must be replaced by their suggested observation time of 60 days, $\delta\phi_2 \sim 2000\,\delta\phi_1$; instead of the improvement by a factor 2×10^4 that they require of their seismic isolation, something like 4×10^7 is needed unless there is to be continuous monitoring and correction of the ellipticity. With the use of conductive damping $\delta\phi_2$ is made negligible, and their specification, already severe enough, is adequate.

Conclusion

The bold proposal by B.P.T. to use a Foucault pendulum at the South Pole as a test of General Relativity was presented by them as extremely difficult but not unrealizable. The above analysis indicates that the neglect of certain disturbing influences puts their proposal, as stated, out of court. On the other hand, the use of parametric excitation and a conductive damping plate appears to restore to the situation something like its original assessment, i.e. an experiment that a superlatively skilled and patient instrument designer might succeed with. Lest, however, any such a one is tempted to engage in so demanding a project, he should ask himself what would happen if he obtained a null result or one differing to some minor degree from the T.L. expression. In view of the large number of disturbing factors to be allowed for, and the absence of any test sites, at least on Earth, where a range of different results could provide corroboration, the response of conservative relativists could not be other than sceptical. The necessary expenditure of time, skill and money should therefore only be incurred for the delight of meeting a challenge, rather than in hope of glory.

I am indebted to Dr D. W. Dewhirst who provided obscure references to early studies of the Foucault pendulum.

After this paper was submitted, Professor Kip Thorne kindly sent me his detailed comments, with helpful suggestions for minor changes. He feels, and I hope he is right, that my assessment of the original B.P.T. proposal is over-pessimistic; at least we agree about its extreme difficulty. I am glad to concur with his view that if the T.L. effect is thought to be worth the inevitably large outlay used to verify it, the Foucault pendulum may well prove the least expensive way of achieving a given degree of precision and credibility.

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